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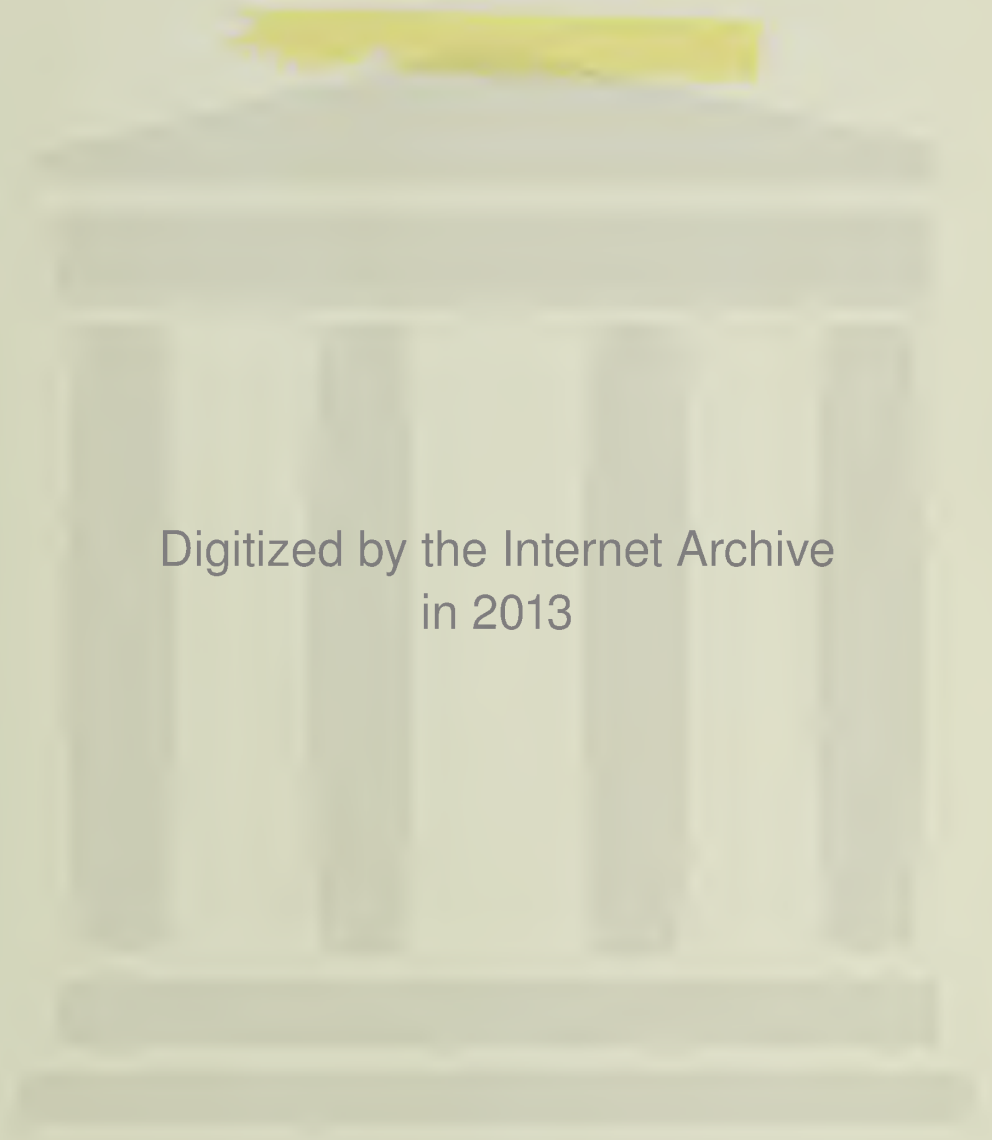
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WEAK CONGRUENCE RELATIONS ON GRAPHS

by

Raymond Tzuu-Yau Yeh

September 1, 1966

DEPARTMENT OF COMPUTER SCIENCE · UNIVERSITY OF ILLINOIS · URBANA, ILLINOIS

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WEAK CONGRUENCE RELATIONS ON GRAPHS*

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Raymond Tzuu-Yau Yeh

September 1, 1966

Department of Computer Science
University of Illinois
Urbana, Illinois 61803

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I. INTRODUCTION

State reduction has been a central problem in the theory of sequential machines. For the complete specified machines, the problem is relatively simple since the states of the reduced machine are congruence classes on the set of states of the original machine. The problem becomes more complicated in the incomplete case because the states of any reduced machine correspond to a cover with substitution property on the states of the original machines as have been shown by many authors [4,5,7,9]. Recently Hartmanis and Stearns pointed out the danger of state reduction in connection with the realizability of a machine by smaller machines. Their investigation [7] showed that sometimes it is desirable to perform "state splitting" before reduction. The concept of "state splitting" seems to tie in with Yoeli's [13] consideration of multi-valued homomorphism quite naturally.

The purpose of this thesis is to investigate the concepts of homomorphic relations (multi-value homomorphisms), weak congruence relations and covers with substitution property on generalized graphs. As the thesis will show these concepts are closely related. In fact, homomorphic relations and weak congruence relations on graphs bear the same natural relations as that of homomorphisms and congruence relations on algebra. More precisely, each weak congruence relation θ on a graph G determines a unique homomorphic relation H_θ on G, and each homomorphic relation H between a graph G and a productive graph G' determines a unique weak congruence relation θ_H on G such that $\theta_H = H \cdot H^T$. Theorems similar to isomorphism theorems on algebra are

obtained. An important theorem in section IV gives conditions under which a finite subcollection of an infinite cover with substitution property on a graph is again a cover with substitution property on the same graph.

It is well known that every sequential machine can be represented by a directed, labelled graph. It is hoped that through the investigation reported here, new insight will be gained in state reduction and realizability of non-deterministic, incomplete sequential machines.

II. ELEMENTARY PROPERTIES OF GRAPHS

Definition 1. Let σ and δ be two relations on a set X, and X_i a subset of X. We define the <u>compositions</u> $\sigma \cdot \delta$ of σ and δ, $X_i \cdot \delta$ of X_i and δ, and $\sigma \cdot X_i$ of σ and X_i by the following rules:

$$\sigma \cdot \delta = \{(x,z) \mid \exists y \in X, (x,y) \in \sigma \wedge (y,z) \in \delta\}$$

$$X_i \cdot \delta = \{y \mid \exists\ x \in X_i, (x,y) \in \delta\}$$

$$\sigma \cdot X_i = \{x \mid \exists\ y \in X_i, (x,y) \in \sigma\}$$

Definition 2. Let X be a set and $\theta \subseteq X \times X$. Then

$I_X = \{(x,x) \mid x \in X\}$ and $\theta^T = \{(x,y) \mid (y,x) \in \theta\}$. θ is <u>reflexive</u> iff $I_X \subseteq \theta$, θ is <u>weakly reflexive</u> iff $\theta \subseteq (\theta \cap I_X) \cdot \theta$.

Remark 1. Let σ, δ and γ be relations on a set X. Then

(1) $\sigma \cdot (\delta \cdot \gamma) = (\sigma \cdot \delta) \cdot \gamma$

(2) $\sigma \cdot (\delta \cup \gamma) = (\sigma \cdot \delta) \cup (\sigma \cdot \gamma)$

(3) $(\sigma \cup \delta) \cdot \gamma = (\sigma \cdot \gamma) \cup (\delta \cdot \gamma)$

(4) $\sigma \cdot (\delta \cap \gamma) \subseteq (\sigma \cdot \delta) \cap (\sigma \cdot \gamma)$

(5) $(\sigma \cap \delta) \cdot \gamma \subseteq (\sigma \cdot \gamma) \cap (\delta \cdot \gamma)$

(6) $(\sigma \cdot \delta)^T = \delta^T \cdot \sigma^T$

(7) $\sigma \subseteq (\sigma \cap I_X) \cdot \sigma \rightleftarrows \forall\ x,y \in X, (x,y) \in \sigma \rightarrow (x,x) \in \sigma$.

Definition 3. Let X and X' be two sets and $\theta \subseteq X \times X'$, then θ is <u>productive</u> iff $\forall$ $x \in X$, there exists at least one $x' \in X'$ such that $(x,x') \in \theta$.

θ is <u>single-valued</u> iff $\forall\, x \in X$, there is exactly one $x' \in X'$ such that $(x,x') \in \theta$.

Remark 2. Let $\theta \subseteq X \times X'$, then

(1) θ is productive iff $\theta \cdot \theta^T \supseteq I_X$

(2) θ is single-valued iff $\theta^T \cdot \theta \subseteq I_{X'}$, and $\theta \cdot \theta^T \supseteq I_X$.

Definition 4. A <u>graph</u> G is a pair $\langle X,A \rangle$, where X is an arbitrary set and A is a finite set of relational symbols such that for each $a \in A$, there corresponds a unique relation $R_a \subseteq X \times X$.

Definition 5. Two graphs $G = \langle X,A \rangle$ and $G' = \langle X',A' \rangle$ are similar iff $A = A'$.

Definition 6. Let $G = \langle X,A \rangle$ and $G' = \langle X',A \rangle$ be two similar graphs. The <u>direct product</u> $G \times G'$ of G and G' is defined by the pair $\langle X \times X',A \rangle$ such that $\forall\, a \in A$, the corresponding relation R_a'' is defined by the rule:

$$\forall\, (x,x'),(y,y') \text{ in } X \times X',\ ((x,x'),(y,y')) \in R_a'' \rightleftarrows (x,y) \in R_a \wedge (x',y') \in R_a' .$$

When $G = G'$, denote $G \times G$ by G^2.

Definition 7. Let $G = \langle X,A \rangle$ and $G' = \langle X',A \rangle$ be two similar graphs. G' is a <u>subgraph</u> of G iff $X' \subseteq X$ and $\forall\, a \in A$, $R_a' = R_a \cap (X' \times X')$.

Remark 3. Let $B = \langle X,A \rangle$ be a graph, then any nonempty subset X' of X determines a unique subgraph of G.

Definition 8. Let $G = \langle X,A \rangle$ be a graph. A relation θ on G satisfies the substitution property (S.P.) iff $\forall\, a \in A$, $R_a^T \cdot \theta \cdot R_a \subseteq \theta$. When θ satisfies S.P., we call θ a S.P. relation.

Remark 4. Let $G = \langle X,A \rangle$ be a graph, θ a relation on G.

(1) θ satisfies S.P. iff

$$\forall\, x,y \in X,\ (x,y) \in \theta \rightarrow (\forall\, a \in A)[\forall\, x' \in x \cdot R_a,\ \forall\, y' \in y \cdot R_a,\ (x',y') \in \theta]$$

(2) θ satisfies S.P. iff $\langle \theta, A \rangle$ is a subgraph of $G^2 = \langle X \times X, A \rangle$.

Let $G = \langle X,A \rangle$ be any graph and $S = \{R_a\}_{a \in A}$. Let S' be the semi-group generated by S under composition, and let $S(G) = S' \cup I$. $\forall\, \alpha \in S(G)$, we define $O(\alpha)$ to be the number of elements of S contained in the minimum representation of α. Let θ be any relation on G, then for each $\alpha \in S(G)$, we define a new relation.

$$\theta_\alpha = \alpha^T \cdot \theta \cdot \alpha$$

Remark 5. Let $G = \langle X,A \rangle$ be a graph such that $\forall\, \alpha \in A$, $R_a \cdot R_a^T = I_X$. Then $\forall\, \alpha \in S(G)$, $\forall\, \theta \subseteq X \times X$, $\forall\, i,j \geq 0$, we have

$(\theta_\alpha)^i = (\theta^i)_\alpha$, where $\theta^i = \theta^{i-1} \cdot \theta$, $i \geq 1$. $\theta^0 = I_X$.

Definition 9. Let θ be a relation on a set X. Define the complement $\overline{\theta}$ of θ by the rule:

$\forall\, x,y \in X,\ (x,y) \in \overline{\theta}$ iff $(x,y) \notin \theta$.

Remark 6. Let σ and δ be two relations on a set X. Then

(1) $\overline{\overline{\sigma}} = \sigma$

(2) $\sigma \subseteq \delta \rightarrow \overline{\delta} \subseteq \overline{\sigma}$

(3) $\overline{\sigma \cup \delta} = \overline{\sigma} \cap \overline{\delta}$

(4) $\overline{\sigma \cap \delta} = \overline{\sigma} \cup \overline{\delta}$

Definition 10. Let $G = \langle X,A \rangle$ be a graph. G is <u>onto</u> iff $\forall\, a \in A$, R_a^T is productive. G is productive iff $\forall\, a \in A$, R_a is productive.

Remark 7. Let $G = \langle X,A \rangle$ be a graph, and $S = \{\sigma_i\}_{i \in I}$ a family of relations on G. Then $\forall\, \alpha \in S(G)$,

(1) $\sigma_i \subseteq \sigma_j \rightarrow (\sigma_i)_\alpha \subseteq (\sigma_j)_\alpha$

(2) $(\sigma_i^T)_\alpha = (\sigma_i)_\alpha^T$

(3) if G is productive, then $(\sigma_i \cdot \sigma_j)_\alpha \subseteq (\sigma_i)_\alpha \cdot (\sigma_j)_\alpha$

(4) $(\bigcup_{i \in I} \sigma_i)_\alpha = \bigcup_{i \in I} (\sigma_i)_\alpha$

(5) $(\bigcap_{i \in I} \sigma_i)_\alpha \subseteq \bigcap_{i \in I} (\sigma_i)_\alpha$

Theorem 1. Let $G = \langle X,A \rangle$ be a graph such that S(G) is abelian. Then $\forall\, \theta \subseteq X \times X$, θ satisfies S.P. iff $\forall\, \alpha \in S(G)$, θ_α satisfies S.P.

Proof: Assume S(G) is abelian and θ satisfies S.P. Then

$$\forall a \in A, \forall \alpha \in S(G), R_a^T \cdot \theta_\alpha \cdot R_a = R_a^T \cdot \alpha^T \cdot \theta \cdot \alpha \cdot R_a$$
$$= \alpha^T \cdot R_a^T \cdot \theta \cdot R_a \cdot \alpha$$
$$\subseteq \alpha^T \cdot \theta \cdot \alpha = \theta_\alpha$$

The converse is clear. Q.E.D.

Lemma 1. Let $G = \langle X, A \rangle$ be a graph, and $\theta \subseteq X \times X$. Then θ satisfies S.P. iff $\forall \alpha \in S(G)$, $\theta_\alpha \subseteq \theta$.

Proof: If $\forall \alpha \in S(G)$, $\theta_\alpha \subseteq \theta$. Then in particular,

$\forall a \in A$, $\theta_a \subseteq \theta \rightarrow R_a^T \cdot \theta \cdot R_a \subseteq \theta$.

Suppose θ satisfies S.P.

$\forall \alpha \in S(G)$, $O(\alpha) \leq 1 \rightarrow \theta_\alpha \subseteq \theta$.

Assume the lemma holds for all α such that $O(\alpha) \leq n$.

Let $\beta \in S(G)$, $O(\beta) = n + 1$. Then $\exists a \in A$, $\alpha \in S(G)$ such that $\beta = \alpha \cdot R_a$ and $O(\alpha) \leq n$.

$$\theta_\beta = \beta^T \cdot \theta \cdot \beta = R_a^T \cdot \alpha^T \cdot \theta \cdot \alpha \cdot R_a$$
$$= R_a^T \cdot \theta_\alpha \cdot R_a$$
$$\subseteq R_a^T \cdot \theta \cdot R_a \subseteq \theta .$$ Q.E.D.

Theorem 2. Let $G = \langle X, A \rangle$ be productive and θ a S.P. relation on G. Then $\forall \alpha \in S(G)$, $\theta_\alpha \subseteq \theta \subseteq \theta_{\alpha^T}$.

Proof: $\forall \alpha \in S(G)$, $\theta_\alpha \subseteq \theta$ by lemma 1.

G is productive $\rightarrow \forall \alpha \in S(G)$,

$$\theta \subseteq \alpha \cdot \alpha^T \cdot \theta \cdot \alpha \cdot \alpha^T = \alpha \cdot \theta_\alpha \cdot \alpha^T$$
$$\subseteq \alpha \cdot \theta \cdot \alpha^T = \theta_{\alpha^T}$$

So, $\theta_\alpha \subseteq \theta \subseteq \theta_{\alpha^T}$. Q.E.D.

Definition 11. Let $G = \langle X,A \rangle$ be a graph and $\theta \subseteq X \times X$.

θ satisfies the dual substitution property (D.S.P.)

iff $\forall a \in A$, $R_a \cdot \theta \cdot R_a^T \subseteq \theta$.

θ is regular if θ satisfies both S.P. and D.S.P.

Theorem 3. Let $G = \langle X,A \rangle$ be a graph and $\theta \subseteq X \times X$.

Then θ satisfies S.P. iff $\overline{\theta}$ satisfies D.S.P.

Proof: Suppose $R_a^T \cdot \theta \cdot R_a \subseteq \theta$.

$\forall x,y \in X$, $(x,y) \in \overline{\theta} \rightleftarrows (x,y) \notin \theta$

$\rightarrow \forall a \in A, \forall x' \in x \cdot R_a^T,$

$\forall y' \in y \cdot R_a^T, (x',y') \notin \theta$

$\rightarrow \forall a \in A, R_a \cdot \overline{\theta} \cdot R_a^T \subseteq \overline{\theta}.$

Converse is proved similarly. Q.E.D.

Definition 12. Let $G = \langle X,A \rangle$ be a graph, define the dual graph $\hat{G}$ of G by the rule:

$\hat{G} = \langle X,A \rangle$ such that $\forall a \in A$, $\hat{R}_a = R_a^T$.

Remark 8. Let $G = \langle X,A \rangle$ be a graph. Then

(1) $\hat{\hat{G}} = G$

(2) G is productive (onto) iff $\hat{G}$ is onto (productive)

(3) $\theta \subseteq X \times X$ satisfies S.P. (D.S.P.) on G iff θ satisfies D.S.P. (S.P.) on $\hat{G}$.

Corollary: Let $G = \langle X,A \rangle$ be an arbitrary graph. Then for each theorem on G involving S.P., there is a corresponding theorem on G involving D.S.P.

Definition 13. A graph G is <u>totally onto</u> iff G is both productive and onto.

Theorem 4. Let $G = \langle X,A \rangle$ be a totally onto graph.

$\theta \subseteq X \times X$ is regular iff $\forall \alpha \in S(G)$, $\theta = \theta_{\alpha} = \theta_{\alpha^T}$.

Proof: θ is regular $\rightleftarrows \forall \alpha \in S(G)$, $\theta_{\alpha} \subseteq \theta \subseteq \theta_{\alpha^T} \wedge \theta_{\alpha^T} \subseteq \theta \subseteq \theta_{\alpha}$

$\rightleftarrows \theta_{\alpha} = \theta = \theta_{\alpha^T}$.

Q.E.D.

III. STRUCTURE OF WEAK CONGRUENCE RELATIONS

Definition 14. Let $G = \langle X,A\rangle$ be a graph and θ a relation on G. Then

(1) θ is a <u>weak congruence relation of type I</u>

(WCRI) iff

(a) $\theta = \theta^T$

(b) $\theta \subseteq (\theta \cap I_X) \cdot \theta$

(c) $\forall\, a \in A,\ R_a^T \cdot \theta \cdot R_a \subseteq \theta.$

(2) θ is a <u>weak congruence relation of type II</u>

(WCRII) iff (b) is replaced by (b') $I_X \subseteq \theta$.

(3) θ is a <u>weak congruence relation of type III</u>

(WCRIII) iff θ is WCRI and (d) $\forall\, a \in A,$

$\theta \cdot R_a \cdot X \subseteq R_a \cdot X.$

(4) θ is a <u>weak congruence relation of type IV</u>

(WCRIV) iff θ is WCRII and satisfies (d).

We note that when G is productive, then WCRI = WCRIII and WCRII = WCRIV.

Definition 15. Let $G = \langle X,A\rangle$ be a graph. A family of nonempty distinct subsets $C = \{X_i\}_{i\in I}$ of X is a <u>cover</u> on G iff $\bigcup_{i\in I} X_i = X$. C is a <u>cover with S.P.</u> on G iff C is a cover on G, and $\forall\ a \in A,\ \forall\, X_i \in C,$ $\exists\, X_j \in C,\ X_i \cdot R_a \subseteq X_j$. C is an <u>irredundant cover</u> on G iff $\forall\, X_i, X_j \in C,\ X_i \subseteq X_j \rightarrow X_i = X_j$.

Let $G = \langle X,A\rangle$ be a graph. Denote the family of all covers on G by C(G). Define an "ordering" "$\leq$" on C(G) such that if $C_i = \{X_j^i\}_{j\in J_i} \in C(G)$, i = 1,2., where J_1 and J_2 are index sets,

then $C_1 \leq C_2 \rightleftarrows \forall X_j^1 \in C_1, \exists X_k^2 \in C_2, X_j^1 \subseteq X_k^2$. When $C_1 \leq C_2$, we say C_1 is a refinement of C_2.

Remark 9. Let $G = \langle X,A \rangle$ be a graph.

(1) "$\leq$" is reflexive and transitive but not anti-symmetric on C(G). However, "$\leq$" is a partial order on the class of all irredundant covers on G.

(2) Every cover (cover with S.P.) $C = \{X_i\}_{i \in I}$ on G defines a unique reflexive and symmetric (WCRII) relation θ_C on G by the rule:
$\forall x,y \in X, (x,y) \in \theta_C \rightleftarrows \exists X_i \in C, x,y \in X_i$.

(3) Every reflexive and symmetric (WCRII) relation θ on T defines a unique cover (cover with S.P.) C_θ on G by the rule:
If X_i is an arbitrary subset of X, then
$X_i \in C_\theta \rightleftarrows$ (i) $\forall x,y \in X_i, (x,y) \in \theta$, and
(ii) $(\forall x \in X)[\forall y \in X_i, (x,y) \in \theta] \rightarrow x \in X_i$.
We note that if θ is not reflexive, then C_θ is simply a family of subsets of X.

(4) There is in general more than one cover (cover with S.P.) on G which determines the same reflexive and symmetric (WCRII) relation on G.

(5) If $C_1, C_2 \in C(G)$, then
$C_1 \leq C_2 \rightleftarrows \theta_{C_1} \subseteq \theta_{C_2}$

(6) If σ and δ are reflexive and symmetric relations on G, then

$\sigma \subseteq \delta \rightleftarrows C_\sigma \leq C_\delta$

(7) If θ is a reflexive and symmetric relation on G, and $C \in C(G)$, then

$\theta_{C_\theta} = \theta$, but $C_{\theta_C} \neq C$ in general.

(8) Mutual refinement of covers is an equivalence relation.

Theorem 5. Let $G = \langle X, A \rangle$ be a graph. Then the partly ordered sets of weak congruence relations of type II and type IV, ordered by refinement, each forms a complete, distributive lattice which is a sublattice of the lattice of all reflexive and symmetric relations on X.

Proof: Let S be any set of weak congruence relations σ on G, then $\alpha = \inf S$ and $\beta = \sup S$ are defined by the rules:

(1) $\alpha = \bigcap_{\sigma \in S} \sigma$

(2) $\beta = \bigcup_{\sigma \in S} \sigma$

It is clear that both α and β are reflexive and symmetric. $\forall$ $a \in A$, we have

$$R_a^T \cdot \alpha \cdot R_a = R_a^T \cdot \bigcap_{\sigma \in S} \sigma \cdot R_a \subseteq \bigcap_{\sigma \in S} R_a^T \cdot \sigma \cdot R_a$$

$$\subseteq \bigcap_{\sigma \in S} \sigma = \alpha$$

$$R_a^T \cdot \beta \cdot R_a = R_a^T \cdot \bigcup_{\sigma \in S} \sigma \cdot R_a = \bigcup_{\sigma \in S} R_a^T \cdot \sigma \cdot R_a = \beta$$

$$(\bigcap_{\sigma\in S}\sigma)\cdot R_a\cdot X\subseteq\bigcap_{\sigma\in S}(\sigma\cdot R_a\cdot X)\subseteq R_a\cdot X$$

$$(\bigcup_{\sigma\in S}\sigma)\cdot R_a\cdot X=\bigcup_{\sigma\in S}(\sigma\cdot R_a\cdot X)\subseteq R_a\cdot X$$

Distributivity follows from the fact that the lattice of all reflexive and symmetric relations on X is a sublattice of the lattice of all subsets of X which is distributive. Q.E.D.

Notation. Let L(G,II) and L(G,IV) be lattices of WCRII and WCRIV respectively on G.

Corollary. Let $G=\langle X,A\rangle$ be a graph, and θ a WCRII on G. Let $L_\theta(G,II)=\{\sigma\,|\,\theta\subseteq\sigma\wedge\sigma\in L(G,II)\}$, and

$$L^\theta(G,II)=\{\sigma\,|\,\sigma\subseteq\theta\wedge\sigma\in L(G,II)\}.$$

Then both $L_\theta(G,II)$ and $L^\theta(G,II)$ are complete sublattices of L(G,II) such that θ is the least element of $L_\theta(G,II)$ and the greatest element of $L^\theta(G,II)$. Similar results hold if θ is WCRIV.

Definition 16. Let $G=\langle X,A\rangle$ be a graph, S a family of nonempty subsets of X. The quotient graph G/S of G with respect to S is defined as follows: $G/S=\langle S,A\rangle$ such that $\forall\, a\in A$, the corresponding relation R'_a is defined such that

$$\forall\, X_i,X_j\in S,\ (X_i,X_j)\in R'_a\rightleftarrows(X_i\cdot R_a\neq\emptyset)\wedge(X_i\cdot R_a\subseteq X_j).$$

If θ is a weakly reflexive and symmetric relation, we define the quotient graph G/θ of G with respect to θ by letting $G/\theta=G/C_\theta$.

Definition 17. Let $G = \langle X, A\rangle$ and $G' = \langle X', A\rangle$ be two similar graphs. Then a relation $H \subseteq X \times X'$ is a homomorphic relation between G and G' iff $\forall\, a \in A$, $R_a^T \cdot H \cdot R_a' \subseteq H$. H is an endomorphic relation on G iff $G = G'$.

Remark 10. Let $G = \langle X, A\rangle$, $G' = \langle X', A\rangle$ and $G'' = \langle X'', A\rangle$ be similar graphs. Then

(1) H is a homomorphic relation between G and G' iff H^T is a homomorphic relation between G' and G.

(2) H is a homomorphic relation between G and G' iff $\forall (x,x') \in X \times X'$,

$$(x,x') \in H \rightarrow (\forall\, a \in A)[\forall\, y \in x \cdot R_a,\ \forall\, y' \in x' \cdot R_a',\ (y,y') \in H]$$

(3) $H \subseteq X \times X$ is an endomorphic relation on G iff H satisfies S.P.

(4) Let $H_1 \subseteq X \times X'$ and $H_2 \subseteq X' \times X''$ be two homomorphic relations between G and G' and between G' and G'' respectively. If G' is productive, then $H_1 \cdot H_2$ is a homomorphic relation between G and G''.

Definition 18. Let X be a set and θ a relation on X. Define trace $T(\theta)$ of θ by the rule:

$$T(\theta) = \begin{cases} 1, & \text{if } \exists\, x \in X,\ (x,x) \in \theta, \\ 0, & \text{if } \forall\, x \in X,\ (x,x) \notin \theta. \end{cases}$$

Remark 11. Let $\{\sigma_i\}_{i\in I}$ be a family of relations on a set X, then

(1) $T(\bigcup_{i\in I} \sigma_i) = \bigcup_{i\in I} T(\sigma_i)$

(2) $T(\sigma_i) = T(\sigma_i^T)$

(3) $T(\sigma_i \cdot \sigma_j) = T(\sigma_j \cdot \sigma_i)$

(4) $\sigma_i \subseteq \sigma_j \rightleftarrows T(\sigma_i \cdot \bar{\sigma}_j^T) = 0.$

Lemma 2. Let $G = \langle X,A\rangle$ and $G' = \langle X',A\rangle$ be two similar graphs, and $H \subseteq X \times X'$. Then

$$\forall a \in A,\ R_a^T \cdot H \cdot R_a' \subseteq H \rightleftarrows R_a' \subseteq \overline{H^T \cdot R_a \cdot \bar{H}}.$$

Proof:

$$R_a^T \cdot H \cdot R_a' \subseteq H \rightleftarrows T(R_a^T \cdot H \cdot R_a' \cdot \bar{H}^T) = 0$$

$$\rightleftarrows T(\bar{H}^T \cdot R_a^T \cdot H \cdot R_a') = 0$$

$$\rightleftarrows \bar{H}^T \cdot R_a^T \cdot H \subseteq \bar{R}_a'^T$$

$$\rightleftarrows H^T \cdot R_a \cdot \bar{H} \subseteq \overline{R_a'}$$

$$\rightleftarrows R_a' \subseteq \overline{H^T \cdot R_a \cdot \bar{H}}$$

Q.E.D.

Theorem 6. Let $G = \langle X,A\rangle$ be a graph and X' an arbitrary set. If $H \subseteq X \times X'$, then there is a graph $G' = \langle X',A\rangle$ such that H is a homomorphic relation between G and G'.

Proof: $\forall a \in A$, define $R_a' \subseteq \overline{H^T \cdot R_a \cdot \bar{H}}$. Theorem follows from lemma 2.

Lemma 3. Let $G = \langle X,A\rangle$ be a graph. Then every family of subsets $S = \{X_i\}_{i\in I}$ of X determines a unique homomorphic relation H_S between G and G/S.

Proof: Define H_S by the relation:

$$\forall x \in X, \forall X_i \in S, (x,X_i) \in H_S \rightleftarrows x \in X_i.$$

$$(x,X_i) \in R_a^T \cdot H_S \cdot R_a' \rightleftarrows \exists y \in X, \exists X_j \in S, (y,x) \in R_a,$$

$$(y,X_j) \in H_S, (X_j,X_i) \in R_a'$$

$$\rightarrow (y \in X_j) \wedge (X_j \cdot R_a \subseteq X_i).$$

Since $x \in y \cdot R_a$, so $x \in X_i$. This implies $(x,X_i) \in H_S$. Thus, $R_a^T \cdot H_S \cdot R_a' \subseteq H_S$.

Q.E.D.

Theorem 7. Let $G = \langle X,A \rangle$ and $G' = \langle X',A \rangle$ be two similar graphs and $H \subseteq X \times X'$ is a homomorphic relation between G and G'. If G' is productive, then H determines a unique weak congruence relation of type I $\theta_H = H \cdot H^T$ on G. Conversely, every WCRI θ on G determines a unique homomorphic relation H_θ between G and G/θ.

Proof: Suppose H is a homomorphic relation on G, and G' is productive. Define $\theta_H = H \cdot H^T$. Then

(1) $\theta_H = \theta_H^T$

(2) $(x,y) \in \theta_H \rightarrow \theta_H \subseteq (\theta_H \cap I_X) \cdot \theta_H$

(3) $\forall a \in A, R_a^T \cdot H \cdot H^T \cdot R_a \subseteq R_a^T \cdot H \cdot R_a' \cdot R_a^T \cdot H^T \cdot R_a$

$$\subseteq H \cdot H^T$$

Thus θ_H is a WCRI on G.

The converse follows from lemma 3.

Q.E.D.

Corollary. Let G and G' be two similar graphs, and H a homomorphic relation between G and G'. If both G' and H are productive, then H determines a unique WCRII.

Definition 19. Let $G = \langle X,A\rangle$ be a graph and σ,γ be two reflexive and symmetric relations on G such that $\sigma \subseteq \gamma$. We define a relation γ/σ on G/σ by the rule:

$$\forall\, x',y' \in G/\sigma,\ (x',y') \in \gamma/\sigma \rightleftarrows \exists\, x \in x' \cdot H_\sigma^T,\ \exists\, y \in y' \cdot H_\sigma^T,\ (x,y) \in \gamma$$

where H_σ is the homomorphic relation H_{C_σ} defined in lemma 3.

Remark 12. Let $G = \langle X,A\rangle$ be a graph and σ,γ two reflexive and symmetric relations on G such that $\sigma \subseteq \gamma$. Then

(1) γ/σ is reflexive and symmetric.

(2) $\gamma/\sigma = H_\sigma^T \cdot \gamma \cdot H_\sigma$.

(3) $\sigma \subseteq \gamma \subseteq \delta \rightarrow \gamma/\sigma \subseteq \delta/\sigma$.

(4) If G is productive and σ,γ are both WCRII, then γ/σ is a WCRII on G/σ.

Definition 20. Let $G = \langle X,A\rangle$ and $G' = \langle X',A\rangle$ be two similar graphs, σ a reflexive and symmetric relation on G', and H a homomorphic relation between G and G', we define $\breve{\sigma}$ on G by the rule:

$$\forall\, x,y \in X,\ (x,y) \in \breve{\sigma} \rightleftarrows \exists\, x' \in x \cdot H,\ \exists\, y' \in y \cdot H,\ (x',y') \in \sigma .$$

Remark 13. Let H be a homomorphic relation between graphs G and G', and σ,δ two reflexive and symmetric relations on G, then

(1) $\breve{\sigma}$ is symmetric

(2) $\breve{\sigma}$ is reflexive iff H is productive

(3) $\sigma \subseteq \gamma \rightarrow \breve{\sigma} \subseteq \breve{\gamma}$

(4) $\breve{\sigma} = H \cdot \sigma \cdot H^{T}$

(5) If both H and G' are productive, then σ is a WCRII on G' implies $\breve{\sigma}$ is a WCRII on G.

Remark 14. Let $G = \langle X,A \rangle$ be a productive graph, σ,γ,δ WCRII on G such that $\sigma \subseteq \gamma \subseteq \delta$. Let θ, be WCRII on G/σ, then

(1) $\gamma \subseteq (\gamma/\breve{\sigma})$

(2) $\theta \subseteq \breve{\theta}/\sigma$

(3) $(\gamma \cup \delta)/\sigma = \gamma/\sigma \cup \delta/\sigma$

(4) $(\gamma \cap \delta)/\sigma \subseteq \gamma/\sigma \cap \delta/\sigma$

(5) $(\theta \breve{\cup} \lambda) = \breve{\theta} \cup \breve{\lambda}$

(6) $(\theta \breve{\cap} \lambda) \subseteq \breve{\theta} \cap \breve{\lambda}$

Theorem 8. Let $G = \langle X,A \rangle$ be a graph. Then to every relation γ on G there corresponds a unique maximum WCRII (WCRIV) $\delta(\gamma)$, and a unique minimum WCRII (WCRIV) $\lambda(\gamma)$ satisfying $\delta(\gamma) \subseteq \gamma \subseteq \lambda(\gamma)$ such that $\delta(\gamma)/\sigma \subseteq \delta(\gamma/\sigma)$, $\lambda(\gamma/\sigma) \subseteq \lambda(\gamma)/\sigma$ if $\sigma \subseteq \gamma$. Furthermore, if S is a set of relations γ on G, then

(1) $\delta(\inf S) = \inf_{\gamma \in S} \{\delta(\gamma)\}$

(2) $\delta(\sup S) = \sup_{\gamma \in S} \{\delta(\gamma)\}$

(3) $\lambda(\inf S) = \inf_{\gamma \in S} \{\lambda(\gamma)\}$

(4) $\lambda(\sup S) = \sup_{\gamma \in S} \{\lambda(\gamma)\}$

Proof: Let C be a family of WCRII (WCRIV) σ on G such that $\sigma \subseteq \gamma$. Then define $\delta(\gamma) = \sup C$. Similarly, define $\lambda(\gamma) = \inf C'$, where C' is the family of all WCRII (WCRIV) σ' on G such that $\gamma \subseteq \sigma'$. Then

$\sigma \subseteq \delta(\gamma) \subseteq \gamma \subseteq \lambda(\gamma) \rightarrow \delta(\gamma)/\sigma \subseteq \gamma/\sigma \subseteq \lambda(\gamma)/\sigma$

$\rightarrow \delta(\gamma)/\sigma \subseteq \delta(\gamma/\sigma)$ and $\lambda(\gamma/\sigma) \subseteq \lambda(\gamma)/\sigma$.

The rest follows from properties of sets.

Q.E.D.

Definition 21. Two relations σ and γ on a set X are commutative with each other iff $\sigma \cdot \gamma = \gamma \cdot \sigma$.

Definition 22. Let $G = \langle X, A \rangle$ be a graph. A family S of pairwise commutative weak congruence relations σ of type k is maximal iff for any weak congruence relation θ of type k on G,

$\theta \cdot \sigma = \sigma \cdot \theta \rightarrow \theta \in S$.

Where k = I, II, III, or IV.

Theorem 9. Let $G = \langle X, A \rangle$ be a productive graph. Then every maximal family of weak congruence relation of type

k, k = I, II, III, and IV is closed under union and composition.

Proof: Let $S = \{\sigma_i\}_{i\in I}$ be a maximal family of weak congruence relations. Then

(1) $[(\sigma_i\cdot\sigma_k = \sigma_k\cdot\sigma_i)\wedge(\sigma_j\cdot\sigma_k = \sigma_k\cdot\sigma_j)] \rightarrow [(\sigma_i\cdot\sigma_j)\cdot\sigma_k = \sigma_k\cdot(\sigma_i\cdot\sigma_j)]$

(2) $[\sigma_i \subseteq (\sigma_i \cap I_X)\cdot\sigma_i,\ i = 1,2] \rightarrow \{(\sigma_1\cdot\sigma_2) \subseteq [(\sigma_1\cdot\sigma_2)\cap I_X]\cdot(\sigma_1\cdot\sigma_2)\}$

(3) $[\sigma_i \supseteq I,\ i = 1,2.] \rightarrow [(\sigma_1\cdot\sigma_2) \supseteq I]$

(4) $[\sigma_i = \sigma_i^T,\ i = 1,2.] \rightarrow [(\sigma_1\cdot\sigma_2)^T = \sigma_2^T\cdot\sigma_1^T = \sigma_1\cdot\sigma_2]$

(5) $\forall\, a \in A,\ R_a^T\cdot\sigma_i\cdot\sigma_j\cdot R_a \subseteq R_a^T\cdot\sigma_i\cdot R_a\cdot R_a^T\cdot\sigma_j\cdot R_a \subseteq \sigma_i\cdot\sigma_j$

(6) G is productive $\rightarrow \forall\, a \in A, \forall\, i \in I,\ \sigma_i\cdot R_a\cdot X \subseteq R_a\cdot X$

(7) $(\sigma_i \cup \sigma_j)\cdot\sigma_k = \sigma_i\cdot\sigma_k \cup \sigma_j\cdot\sigma_k = \sigma_k\cdot(\sigma_i \cup \sigma_j)$

Q.E.D.

Theorem 10. Let $G = \langle X,A\rangle$ be a totally onto graph, and $S(G)$ abelian. Then θ is a weak congruence relation of type k on G iff $\forall\, \alpha \in S(G)$, θ_α is a weak congruence relation of type k, where k = I, II, III, or IV.

Proof: In view of theorm 1, it is suffice to show that

(1) $\forall\, \alpha \in S(G),\ (\alpha^T\cdot\theta\cdot\alpha)^T = \alpha^T\cdot\theta\cdot\alpha \rightarrow \theta_\alpha^T = \theta_\alpha$

(2) $\theta \subseteq (\theta \cap I_X) \cdot \theta \rightarrow \forall \alpha \in S(G),\ \theta_\alpha = \alpha^T \cdot \theta \cdot \alpha$

$\subseteq \alpha^T \cdot (\theta \cap I_X) \cdot \theta \cdot \alpha$

$\subseteq \alpha^T \cdot (\theta \cap I_X) \cdot \alpha \cdot \alpha^T \cdot \theta \cdot \alpha$

$\subseteq (\theta_\alpha \cap I_X) \cdot \theta_\alpha$

(3) $I \subseteq \theta \rightarrow \forall \alpha \in S(G),\ (\alpha^T \cdot \theta \cdot \alpha)$

$\supseteq \alpha^T \cdot I \cdot \alpha = \alpha^T \cdot \alpha \supseteq I$

(4) G is productive $\rightarrow \forall a \in A,\ \theta \cdot R_a \cdot X \subseteq R_a \cdot X$

Q.E.D.

Definition 23. Let $G = \langle X, A \rangle$ be a graph and θ any relation on G. The <u>closure</u> θ^* of θ is defined by the rule:

$$\theta^* = \bigcup_{\alpha \in S(G)} \theta_\alpha$$

Theorem 11. Let $G = \langle X, A \rangle$ be a graph and θ, σ, δ be relations on G. Then

(1) θ^* is the smallest S.P. relation which contains θ.

(2) $\forall \alpha \in S(G),\ \theta_\alpha \subseteq \theta^*$

(3) $\sigma \subseteq \delta \rightarrow \sigma^* \subseteq \delta^*$

(4) $(\theta^*)^* = \theta^*$

Proof: $\forall a \in A,\ R_a^T \cdot \theta^* \cdot R_a = R_a^T \cdot \bigcup_{\alpha \in S(G)} \theta_\alpha \cdot R_a$

$= \bigcup_{\alpha \in S(G)} R_a^T \cdot \theta_\alpha \cdot R_a$

$= \bigcup_{\alpha \in S(G)} R_a^T \cdot \alpha^T \cdot \theta \cdot \alpha \cdot R_a \subseteq \theta^*$

$\sigma \subseteq \delta \rightarrow \forall \alpha \in S(G),\ \sigma_\alpha \subseteq \delta_\alpha \rightarrow \sigma^* \subseteq \delta^*$

$\theta \subseteq \theta^* \rightarrow \theta^* \subseteq (\theta^*)^*$

$$\forall a \in A,\ R_a^T \cdot \theta^* \cdot R_a \subseteq \theta^* \rightarrow \forall \alpha \in S(G),\ (\theta^*)_\alpha \subseteq \theta^* \rightarrow (\theta^*)^* \subseteq \theta^*$$

Thus, $(\theta^*)^* = \theta^*$

Suppose γ is a S.P. relation on G such that $\theta \subseteq \gamma$.

We see $\theta_\alpha \subseteq \gamma$ for $O(\alpha) = 0$.

Assume $\theta_\alpha \subseteq \gamma$, $\forall \alpha \in S(G)$ such that $O(\alpha) \leq n$.

Let $\beta \in S(G)$, $O(\beta) = n + 1$. Then $\beta = \alpha \cdot R_a$ for some $a \in A$, and some $\alpha \in S(G)$ such that $O(\alpha) \leq n$.

$$\theta_\beta = \beta^T \cdot \theta \cdot \beta = R_a^T \cdot \alpha^T \cdot \theta \cdot \alpha \cdot R_a \subseteq R_a^T \cdot \gamma \cdot R_a \subseteq \gamma \rightarrow \theta^* \subseteq \gamma.$$

Q.E.D.

Definition 24. Let $G = \langle X, A \rangle$ be a graph and θ a relation on G. Define the <u>reflexive extension $R(\theta)$</u> and <u>symmetric extension $S(\theta)$</u> of θ by the rule:

$R(\theta) = \theta \cup I$, $S(\theta) = \theta \cup \theta^T$.

Thus, we see $RS(\theta) = SR(\theta) = \theta \cup \theta^T \cup I$.

Theorem 12. Let $G = \langle X, A \rangle$ be a graph and θ a relation on G. Then $\lambda(\theta) = RS(\theta)^*$.

Proof: If σ is any WCRII on G such that $\theta \subseteq \sigma$, then $RS(\theta) \subseteq \lambda(\theta) \subseteq \sigma$. By theorem 11, $RS(\theta)^* \subseteq \sigma$. In particular, $RS(\theta)^* \subseteq \lambda(\theta)$. However, $RS(\theta)^*$ is a WCRII containing θ. So, $\lambda(\theta) \subseteq RS(\theta)^*$. Thus, $\lambda(\theta) = RS(\theta)^*$.

Q.E.D.

Definition 25. Let $G = \langle X,A \rangle$ be a graph and θ a relation on G. θ is a dual weak congruence relation of type k iff θ satisfies all the conditions, except S.P., for a weak congruence relation of type k and D.S.P., where k = I, II, III, or IV.

Remark 15. Let $G = \langle X,A \rangle$ be a graph and θ a relation on G. Let $\delta_d(\theta)$ and $\lambda_d(\theta)$ be the unique maximum dual weak congruence relation of type II and unique minimum dual weak congruence relation of type II on G satisfying $\delta_d(\theta) \subseteq \theta \subseteq \lambda_d(\theta)$. If we let $\hat{\theta} = \overline{\theta} \cup \overline{\theta}^T \cup I$, then we have

(1) θ is reflexive and symmetric $\rightarrow \hat{\hat{\theta}} = \theta$

(2) $\sigma \subseteq \delta \rightarrow \hat{\delta} \subseteq \hat{\sigma}$

(3) σ is WCRII and $\sigma \subseteq \theta$ implies that $\hat{\sigma}$ is a dual weak congruence relation of type II such that $\hat{\theta} \subseteq \hat{\sigma}$.

Theorem 13. Let $G = \langle X,A \rangle$ be a graph and θ a relation on G. Then $\delta(\theta) = \lambda_d(\hat{\theta})$.

Proof: $\lambda_d(\hat{\theta})$ is reflexive and symmetric by definition.

$\lambda_d(\hat{\theta})$ satisfies D.S.P. implies $\lambda_d(\hat{\theta})$ satisfies S.P.

Thus $\lambda_d(\hat{\theta})$ is a WCRII.

$\hat{\theta} \subseteq \lambda_d(\hat{\theta}) \rightarrow \lambda_d(\hat{\theta}) \subseteq \hat{\hat{\theta}} = \theta$.

$\lambda_d(\hat{\theta}) \subseteq \sigma$ for all dual weak congruence relation σ of type II such that $\hat{\theta} \subseteq \sigma$. Thus, $\sigma \subseteq \lambda_d(\hat{\theta})$. Since

$\delta(\theta) \subseteq \theta$ and $\delta(\theta) = \hat{\delta}(\theta)$, so $\delta(\theta) \subseteq \lambda_d\hat{(\theta)}$.

But $\lambda_d\hat{(\theta)} \subseteq \theta \rightarrow \lambda_d\hat{(\theta)} \subseteq \delta(\theta)$.

So, $\delta(\theta) = \lambda_d\hat{(\theta)}$.

Q.E.D.

Definition 26. Let $G = \langle X,A\rangle$ and $G' = \langle X',A\rangle$ be two similar graphs. Then $H \subseteq X \times X'$ is a strong homomorphic relation between G and G' iff $\forall a \in A$,

(i) $R_a^T \cdot H \cdot R_a \subseteq H$,

(ii) $H^T \cdot R_a \cdot X \subseteq R'_a \cdot X'$,

(iii) $H \cdot R'_a \cdot X' \subseteq R_a \cdot X$.

Remark 16.

(1) H is a strong homomorphic relation between G and G' iff H^T is a strong homomorphic relation between G' and G.

(2) Let $G = \langle X,A\rangle$, $G' = \langle X',A\rangle$ and $G'' = \langle X'',A\rangle$ be similar graphs. If $H \subseteq X \times X'$ and $H' \subseteq X' \times X''$ are strong homomorphic relations between G and G', and between G' and G'' respectively, then $H \cdot H' \subseteq X \times X''$ is a strong homomorphic relation between G and G''.

Theorem 14. Let $G = \langle X,A\rangle$ be a graph and $\theta \subseteq X \times X$. θ is a weak congruence relation of type III on G iff there exist a graph $G' = \langle X',A\rangle$ and a strong homomorphic relation H between G and G' such that $\theta = H \cdot H^T$.

Proof: Suppose H is a strong homomorphic relation between G and G', and $\theta = H \cdot H^T$. It is clear that $\theta = \theta^T$, and $\theta \subseteq (\theta \cap I_X) \cdot \theta$.

$\forall a \in A$, $H^T \cdot R_a \cdot X \subseteq R'_a \cdot X' \rightarrow H \cdot H^T \cdot R_a \cdot X \subseteq H \cdot R'_a \cdot X' \subseteq R_a \cdot X$.

I.e., $\theta \cdot R_a \cdot X \subseteq R_a \cdot X$.

$\forall a \in A$, $\forall x,y \in X$, $(x,y) \in R_a^T \cdot \theta \cdot R_a \rightleftarrows (x,y) \in R_a^T \cdot H \cdot H^T \cdot R_a$

$\rightarrow \exists w \in X, \exists w' \in X', \exists z \in X, x\ R_a^T\ w\ H\ w'\ H^T\ z\ R_a\ y$

$\rightarrow \exists \mu' \in X', x\ R_a^T\ w\ Hw'R'_a\ \mu'R_a'^T\ w'H^Tz\ R_a\ y$

since $H^T \cdot R_a \cdot X \subseteq R'_a \cdot X'$

$\rightarrow (x,y) \in R_a^T \cdot H \cdot R'_a \cdot R_a'^T \cdot H^T \cdot R_a \subseteq H \cdot H^T = \theta$

So, $\forall a \in A$, $R_a^T \cdot \theta \cdot R_a \subseteq \theta$.

We have shown that θ is a weak congruence relation of type III.

We now suppose that θ is a weak congruence relation of type III on G.

Define $G' = \langle X', A \rangle$ by the rules:

(i) $X' = C_\theta$, and

(ii) $\forall a \in A$, $\forall X_i, X_j \in C_\theta$, $(X_i, X_j) \in R'_a$

$\rightleftarrows (X_i \cdot R_a \neq \emptyset) \wedge (X_i \cdot R_a \subseteq X_j)$.

Define $H \subseteq X \times X'$ by the rule:

$\forall x \in X$, $\forall X_i \in C_\theta$, $(x, X_i) \in H \rightleftarrows x \in X_i$.

It is clear that $\theta = H \cdot H^T$.

$\forall a \in A$, $(x,X_i) \in R_a^T \cdot H \cdot R_a' \rightleftarrows \exists y \in X$, $\exists X_j \in C_\theta$,

$x R_a^T y$ H $X_j R_a' X_i$.

$(y,X_j) \in H \rightleftarrows y \in X_j \rightarrow \forall y' \in y \cdot R_a$, $y' \in X_i$,

since $(X_j,X_i) \in R_a'$.

$\rightarrow x \subset X_i \rightarrow (x,X_i) \in H \rightarrow \forall a \in A$, $R_a^T \cdot H \cdot R_a' \subseteq H$.

$X_i \in H^T \cdot R_a \cdot X \rightleftarrows \exists x \in X$, $(x,X_i) \in H$ and $x \cdot R_a \neq \emptyset$

$\rightarrow X_i \cdot R_a \neq \emptyset \rightarrow \exists X_j \in C_\theta$, $X_i \cdot R_a \subseteq X_j$,

since C_θ satisfies

S.P.

$\rightarrow \forall a \in A$, $H^T \cdot R_a \cdot X \subseteq R_a' \cdot X'$.

$x \in H \cdot R_a' \cdot X' \rightleftarrows \exists X_i \in X'$, $(x,X_i) \in H$ and $X_i \cdot R_a' \neq \emptyset$

$\rightarrow \exists y \in X_i$, $y \cdot R_a \neq \emptyset$

$\rightarrow x \cdot R_a \neq \emptyset$, since $(x,y) \in \theta$,

and $\theta \cdot R_a \cdot X \subseteq R_a \cdot X$.

$\rightarrow \forall a \in A$, $H \cdot R_a' \cdot X' \subseteq R_a \cdot X$.

We have shown that there exist a graph G' and a strong homomorphic relation H between G and G' such that $\theta = H \cdot H^T$.

Q.E.D.

Definition 27. Let $G = \langle X,A \rangle$ and $G' = \langle X',A \rangle$ be two similar graphs. $H \subseteq X \times X'$ is an <u>isomorphic relation</u> between G and G' iff H is a strong homomorphic relation between G and G' such that both H and H^T are productive.

Notation: We will use the symbols $G \sim G'$ to denote the fact that there is an isomorphic relation between G and G'.

Theorem 15. Let H be a strong homomorphic relation between two graphs $G = \langle X,A \rangle$ and $G' = \langle X',A \rangle$. Then $G/\theta_H \sim \langle X \cdot H,A \rangle$.

Proof: $\theta_H = H \cdot H^T$ is a weak congruence relation of type III by theorem 14. By the same theorem, there exists a strong homomorphic relation H' between $\langle H \cdot X',A \rangle$ and G/θ_H such that H' and H'^T are both productive. Define $\hat{H} \subseteq C_{\theta_H} \times (X \cdot H)$ by $\hat{H} = H'^T \cdot H$. Then $\hat{H}$ is a strong homomorphic relation. Moreover, both $\hat{H}$ and $\hat{H}^T$ are productive.

So, $G/\theta_H \sim \langle X \cdot H,A \rangle$.

Q.E.D.

Definition 28. Let $G = \langle X,A \rangle$ be a graph. $G' = \langle X',A \rangle$ is a <u>semi-productive</u> subgraph of G iff G' is a subgraph of G, and $\forall\ x \in X',\ x \cdot R_a \neq \emptyset \rightarrow x \cdot R'_a \neq \emptyset$

Theorem 16. Let $G_1 = \langle X_1,A \rangle$ be a graph, $G_2 = \langle X_2,A \rangle$ a semi-productive subgraph of G_1, and θ a weak congruence relation of type III on G_1, then $G_2/\theta \cap (X_2 \times X_2) \sim \langle X_2 \cdot \theta,A \rangle/\theta$.

Proof: θ determines a strong homomorphic relation H between G_1 and G_1/θ by theorem 14. Let $\hat{H}$ be the relation H restricted to X_2. We see then $\hat{H}$ is a strong homomorphic relation between G_2 and $\langle X_2 \cdot \theta,A \rangle/\theta$ defined

by the rule:

$$\forall\, x \in X_2,\ x \cdot \hat{H} = (x \cdot \theta)/\theta \ .$$

Both $\hat{H}$ and $\hat{H}^T$ are productive, and $\hat{H} \cdot \hat{H}^T = \theta \cap (X_2 x X_2)$.

Thus, by theorem 15, we have

$$G_2/\theta \cap (X_2 x X_2) \sim \langle X_2 \cdot \theta, A\rangle/\theta \ .$$

Q.E.D.

Theorem 17. Let $G = \langle X, A\rangle$ be a productive graph, and σ, δ two weak congruence relations of type II on G such that $\sigma \subseteq \delta$. Then $(G/\sigma)/(\delta/\sigma) \sim G/\delta$.

Proof: Let $C_\sigma = \{X_i\}_{i\in I}$ and $C_\delta = \{Y_j\}_{j\in J}$.

Define $H \subseteq C_\sigma \times C_\delta$ by the rule:

$$\forall\, X_i \in C_\sigma,\ \forall\, Y_j \in C_\delta,\ (X_i Y_j) \in H \rightleftarrows X_i \cap Y_j \neq \emptyset \ .$$

$\forall\, a \in A$, let R'_a and R''_a be the corresponding relations in G/σ and G/δ respectively.

$$\forall\ a \in A,\ \forall\, X_i \in C_\sigma,\ \forall\, Y_j \in C_\delta,$$

$$X_i\, R_a'^{T} \cdot H \cdot R''_a\, Y_j \rightleftarrows \exists\, X'_i,\ \exists\, Y'_j,\ X_i R_a'^{T} X'_i\ H\ Y'_j R''_a\, Y_j$$

$$\rightarrow X'_i \cap Y'_j = \emptyset$$

G is productive $\rightarrow$ G/σ and G/δ are productive.

So, $\forall\, X''_i \in X'_i \cdot R'_a,\ \forall\, Y''_j \in Y'_j \cdot R''_a,\ X''_i \cap Y''_j \neq \emptyset$

In particular, $X_i \cap Y_j \neq \emptyset$. So $(X_i, Y_j) \in H$

Thus, $\forall\, a \in A,\ R_a'^{T} \cdot H \cdot R''_a \subseteq H$.

G/σ and G/δ are both productive imply that $\forall\, a \in A$,

$H^T \cdot R'_a \cdot C_\sigma \subseteq R''_a \cdot C_\delta$ and $H \cdot R''_a \cdot C_\delta \subseteq R'_a \cdot C_\sigma$

Therefore, H is a strong homomorphic relation between G/σ and G/δ. By theorem 16, $(G/\sigma)/\theta_H \sim G/\delta$, where $\theta_H = H \cdot H^T$.

We must show that $\theta_H = \delta/\sigma$.

$$\forall X_i, X_j \in C_\sigma,\ (X_i, X_j) \in \theta_H \rightleftarrows X_i \cdot H \cap X_j \cdot H \neq \emptyset$$

$$\rightleftarrows \exists\ Y_k \in C_\sigma,$$

$$X_i \cap Y_k \neq \emptyset \wedge X_j \cap Y_k \neq \emptyset$$

$$\rightleftarrows \exists\ x \in X_i,\ \exists\ y \in X_j,\ (x,y) \in \delta$$

$$\rightleftarrows (X_i, X_j) \in \delta/\sigma$$

$$\rightarrow \theta_H = \delta/\sigma.$$

Therefore, $(G/\sigma)/(\delta/\sigma) \sim G/\delta$

Q.E.D.

IV. CANONICAL AND FINITELY REDUCIBLE COVERS

Definition 29. A cover with S.P. $C = \{X_i\}_{i\in I}$ on a graph $G = \langle X,A\rangle$ is canonical iff

$$\forall\, a \in A,\ \forall\, X_i \in C,\ X_i \cdot R_a \neq \emptyset \rightarrow X_i \cdot R_a \in C.$$

Theorem 18. Let $G = \langle X,A\rangle$ be a graph. Then to every cover with S.P. $C = \{X_i\}_{i\in I}$ on G, there corresponds a canonical cover with S.P. $\hat{C}$ on G such that $\theta_C = \theta_{\hat{C}}$.

Proof: Let C be a given cover with S.P.

Define $C_0 = C_{\theta_C} = \{X_j^{(0)}\}_{j\in J_0}$

After $C_n = \{X_j^{(n)}\}_{j\in J}$ has been defined, $n \geq 0$,

we define $C_{n+1} = \left\{X_j^{(n)} \cdot R_a \,\middle|\, a\in A,\ j\in J_n,\ X_j^n \cdot R_a \neq \emptyset\right\}$

$$= \left\{X_j^{(n+1)}\right\}_{j\in J_{n+1}}.$$

Define $\hat{C} = \left\{X_j^{(k)} \,\middle|\, j \in \bigcup_{i=0}^{\infty} J_i,\ k \geq 0\right\}$

It is clear that $\hat{C}$ is a canonical cover such that $\theta_C = \theta_{\hat{C}}$.

Q.E.D.

Corollary: If C_i, $i = 1,2$ are two covers with S.P. on a graph G, then $\theta_{C_1} = \theta_{C_2} \rightarrow \hat{C}_1 = \hat{C}_2$.

Theorem 19. Let $G = \langle X, A \rangle$ be an infinite graph such that $A = \{a\}$. Then a finite canonical cover with S.P. on X exists iff $\exists N \geq 0$, $X \cdot R_a^{N+1} = X \cdot R_a^N$. Where $X \cdot R_a^0 = X$ and $X \cdot R_a^{n+1} = X \cdot R_a^n \cdot R_a$.

Proof: Suppose $\exists N \geq 0$, $X \cdot R_a^{N+1} = X \cdot R_a^N$.

Then $C = \left\{X \cdot R_a^i\right\}_{i=0}^{N}$ is a finite canonical cover with S.P. on G.

Suppose now that $C = \left\{X_i\right\}_{i \in I}$ is a finite canonical cover with S.P. on G. Then

$$X \cdot R_a = \left(\bigcup_{i \in I} X_i\right) \cdot R_a = \bigcup_{i \in I} X_i \cdot R_a$$

Since C is canonical, $\forall X_i \in C$, $X_i \cdot R_a \neq \emptyset \rightarrow X_i \cdot R_a \in C$.

Therefore, $X \cdot R_a = \bigcup_{i \in I^{(1)}} X_i$, where $I^{(1)} \subseteq I$.

$$X \cdot R_a^n = X \cdot R_a^{n-1} \cdot R_a = \bigcup_{i \in I^{(n-1)}} X_i \cdot R_a = \bigcup_{i \in I^{(n)}} X_i .$$

Since I is a finite set, there must exist an $N \geq 0$ such that

$$I \supset I^{(1)} \supset I^{(2)} \supset \ldots\ldots \supset I^{(N)} = I^{(N+1)} .$$

Since $X \cdot R_a^N = \bigcup_{i \in I^{(N)}} X_i$

So, $\exists N \geq 0$, $X \cdot R_a^{N+1} = X \cdot R_a^N$.

Q.E.D.

Definition 30. Let $G = \langle X,A \rangle$ be a graph. An element $x_0 \in X$ is a sequential vertex for a subset S of X iff there is an infinite sequence of distinct elements $\{X_i\}_{i=0}^{\infty}$ such that

$$[\forall i \geq 0, \exists a \in A, (x_i, x_{i+1}) \in R_a] \wedge [\exists N \geq 0, \forall n > N, x_n \in S]$$

Definition 31. Let $G = \langle X,A \rangle$ be a graph and S a subset of X. S is closed if it contains all the sequential vertices for itself.

Lemma 4. Let $G = \langle X,A \rangle$ be a graph. Then

(1) all finite subsets of X are closed.

(2) the intersection of an arbitrary family of closed subsets of X is closed.

Proof:

(1) The set of sequential vertices for any finite subset of X is empty.

(2) Let $\{X_i\}_{i \in I}$ be an arbitrary family of closed subsets of X. Let $S = \bigcap_{i \in I} X_i$. If x is a sequential vertex for S, then x is a sequential vertex for X_i, $i \in I$. This implies that $x \in X_i$, $i \in I$. I.e., $x \in S$.

So, S is closed.

Q.E.D.

Definition 32. A graph $G = \langle X,A \rangle$ is sequential if every infinite subset X_i of X has a sequential vertex in X.

Remark 17. A graph $G = \langle X,A \rangle$ is sequential iff every infinite subgraph of G is sequential.

Lemma 5. Let $\{X_k\}_{k=1}^{\infty}$ be a countably infinite collection of nonempty subsets of X of a graph $G = \langle X,A \rangle$ such that

(1) $X_{k+1} \subset X_k$, $\forall\, k > 0$, and

(2) each X_k is closed and X_1 is sequential.

Then $\bigcap_{k=1}^{\infty} X_k$ is closed and nonempty.

Proof: $\bigcap_{k=1}^{\infty} X_k$ is closed by lemma 4.

By the condition of the lemma, each X_k must be infinite. Therefore, we can form a collection of distinct points $S = \{x_i\}_{i=1}^{\infty}$ such that $x_i \in X_i$. S is an infinite subset of X_1 which is sequential, therefore, there exists a sequential vertex y_0 for S such that $y_0 \in X_1$. I.e. there is an infinite sequence of distinct elements $\{y_i\}_{i=0}^{\infty}$ such that $[\forall\, i \geq 0, \exists\, a \in A, (y_i,y_{i+1}) \in R_a] \wedge [\exists\, N \geq 0, \forall\, n > N, y_n \in S]$. Let $S' = \{y_i\}_{i=N+1}^{\infty}$.

Since all but a finite number of elements of S belong to X_k, $\forall\, k > 0$, S' must also contain an infinite number of elements of X_k. I.e., $\exists\, N_k \geq N+1$, $\forall\, m > N_k$, $y_m \in X_k$. Therefore, y_0 is a sequential vertex for X_k, $\forall\, k > 0$. Since $\forall\, k > 0$, X_k is closed,

so $y_0 \in X_k$, $\forall\ k > 0$.

I.e., $y_0 \in \bigcap_{k=1}^{\infty} X_k \rightarrow \bigcap_{k=1}^{\infty} X_k \neq \emptyset$.

Q.E.D.

Definition 33. Let $B = \langle X,A \rangle$ be a graph, and C a cover with S.P. on G. C is <u>finitely reducible</u> iff there is a finite subcollection of elements of C which is again a cover with S.P. on G.

Definition 34. Let $G = \langle X,A \rangle$ be a graph. G is <u>semi-cyclic</u> iff $\forall\ x \in X$, there is a finite sequence $a_1\ a_2 \cdots a_{n(x)}$ of elements of A such that $(x,x) \in \prod_{i=1}^{n(x)} R_{a_i}$.

Definition 35. Let $G = \langle X,A \rangle$ be a graph and C a cover with S.P. on G. C is a semi-cyclic cover on G iff G/C is semi-cyclic.

Theorem 20. If C is a semi-cyclic cover with S.P. on a sequential graph $G = \langle X,A \rangle$. Then C is finitely reducible.

Proof: If C is finite, nothing to prove. So,

assume $C = \{X_i\}_{i=1}^{\infty}$.

$\forall\ m \geq 1$, consider the finite subcollection $S_m = \{X_i\}_{i=1}^{m}$ of C. We claim that there exists an $N > 0$ such that S_N is a cover on G. Suppose there does not exist such an N. Now $\forall\ i > 0$, $X - X_i$ is a closed subset of X. Since if $y_0 \in X_i$ and y_0 is a

sequential vertex for $X - X_i$, then let $\{y_i\}_{i=0}^{\infty}$ be the infinite sequence determined by y_0. Because C is semi-cyclic, an infinite number of elements of X_i will occur in this sequence. This contradicts the fact that y_0 is a sequential vertex for $X - X_i$. Thus, $X - X_i$ cannot have a sequential vertex outside of itself and therefore must be closed.

Now let $Q_0 = X$, and $\forall\, i > 0$, $Q_i = X - \bigcup_{X_j \epsilon S_i} X_j$.

So, $Q_i = \bigcap_{X_j \epsilon S_i} (X - X_j)$, $\forall\, i > 0$.

Since $X - X_j$ is closed $\forall\, j > 0$, Q_i is also closed $\forall\, i > 0$. Furthermore, $\forall\, i > 0$, $Q_{i+1} \subset Q_i$ by the hypothesis that S_N is not a cover of X, $\forall\, N > 0$. Thus, we have a sequence of closed subsets $\{Q_i\}_{i=0}^{\infty}$ satisfying the hypothesis of lemma 5. Hence, by lemma 5, $\bigcap_{i=0}^{\infty} Q_i \neq \emptyset$. However, we note that $\forall\, x \in X$, $x \in Q_i \rightarrow x \notin \bigcup_{X_j \epsilon S_i} X_j$. Therefore,

$$x \in \bigcap_{i=0}^{\infty} Q_i \rightarrow x \notin \bigcup_{j=1}^{\infty} X_j \rightarrow x \notin X \rightarrow C \text{ is not a cover on X.}$$

We thus arrive at a contradiction. Hence, $\exists$ N such that $\bigcup_{i=1}^{N} X_i = X$. Since C is semi-cyclic, $\forall\, X_i \in C$, there is a finite sequence $a_1, \ldots, a_{n(X_i)}$ of elements of A such that $(X_i, X_i) \in \prod_{k=1}^{n(X_i)} R_{a_k}$.

Thus, we can construct the following sets:

$$X_1 = X_{10}, X_{11}, \cdots, X_{1,n(X_1)-1}$$
$$\vdots$$
$$X_N = X_{N0}, X_{N1}, \cdots, X_{N,n(X_N)-1}$$

Where $\forall\ 1 \leq i \leq N$, $\forall\ 0 \leq j < n(X_i)-1$, $\exists\ a \in A$, $(X_{ij}, X_{i\ j+1}) \in R'_a$. And $(X_{i\ n(X_i)-1}, X_i) \in R'_b$, some $b \in A$. Where $\forall\ a \in A$, R'_a is the corresponding relation on G/C.

$$\text{Let } C^* = \left\{ X_{ij} \right\}_{\substack{1 \leq i \leq N \\ 0 \leq j \leq n(X_i)-1}}$$

It is clear that C^* is a cover with S.P. on G. Furthermore, C^* is a finite subcollection of C. So, C is finitely reducible.

Q.E.D.

Definition 36. Let $G = \langle X, A \rangle$ be a graph. A <u>path</u> $P(x,y)$ from x to y on G is a finite sequence $x = x_0, x_1, \ldots, x_n = y$ of elements of X such that $\forall\ 0 \leq i \leq n-1$, $\exists\ a \in A$, $(x_i, x_{i+1}) \in R_a$. We will denote by $S_{P(x,y)}$ the set of all elements of X occur in the path $P(x,y)$.

Definition 37. A graph $G = \langle X, A \rangle$ is <u>semi-connected</u> iff for each infinite subset X' of X, there exists an $x \in X'$ and an infinite subset X'' of X' such that $\forall\ y \in X''$, there is a path $P(x,y)$ and $S_{P(x,y)} \subseteq X'$.

Remark 18. A graph $G = \langle X,A \rangle$ is semi-connected iff every infinite subgraph of G is semi-connected.

Theorem 21. Let $G = \langle X,A \rangle$ be an infinite graph. Then G is sequential iff G is semi-connected.

Proof: Suppose G is sequential.

Let X' be an arbitrary infinite subset of X. Then there is a sequential vertex $x_0 \in X$ for X'. Let $\{X_i\}_{i=0}^{\infty}$ be the infinite sequence determined by x_0, we see that $\exists\, N \geq 0$, $\forall\, n > N$, $x_n \in X'$. Let $X'' = \{x_i\}_{i=N+1}^{\infty}$. Then $\forall\, x_i \in X''$, $\exists\, P(x_N,x_i)$ and $S_{P(x_N,x_i)} \subseteq X$. X' arbitrary implies G is semi-connected. Suppose now G is semi-connected. Let X_1 be an arbitrary infinite subset of X. Then $\exists$ infinite subset $X_2 \subseteq X_1$, such that $\forall\, y \in X_2$, $\exists\, P(x_1,y) \wedge S_{P(x_1,y)} \subseteq X_1$. Let $X_2' = X_2 - x_1$. Then X_2' is again an infinite subset of X and hence is semi-connected. Therefore, $\exists\, x_2 \in X_2'$, $\exists$ infinite subset $X_3 \subseteq X_2'$ such that $\forall\, y \in X_3$, $\exists\, P(x_2,y) \wedge S_{P(x_2,y)} \subseteq X_2'$. We may assume that $P(x_1,x_2)$ is a path consisting of distinct elements (by eliminating all loops). Let $X_3' = X_3 - S_{P(x_1,x_2)}$. Then X_3' is again an infinite subset of X. We then proceed in the same manner to obtain x_3. After we have obtained x_n and the infinite subset X_{n+1}, $n > 1$,

let $X'_{n+1} = X_{n+1} - S_{P(x_{n-1},x_n)}$, where we may assume $S_{P(x_{n-1},x_n)}$ consists of distinct elements. X'_{n+1} is an infinite subset of X implies $\exists\ x_{n+1} \in X'_{n+1}$, $\exists$ infinite subset $X_{n+2} \subseteq X'_{n+1}$ such that $\forall\ y \in X_{n+2}$, $\exists\ P(x_{n+1},y)$ and $S_{P(x_{n+1},y)} \subseteq X'_{n+1}$. Hence we obtain x_{n+1}. Proceed in this way, we obtain an infinite sequence $\{x_i\}_{i=1}^{\infty}$ such that $\forall\ i > 0$, $\exists\ P(x_i,x_{i+1})$ and $S_{P(x_i,x_{i+1})} \subseteq X_1$. Now, enlarge this sequence by adding all the intermediate elements between x_i and x_{i+1} in the path $P(x_i,x_{i+1})$ $\forall\ i > 0$. Then the resulting enlarged sequence $\{x'_i\}_{i=0}^{\infty}$ has the properties (1) $x'_0 = x_1$, (2) all elements in the sequence are distinct, (3) $\forall\ i \geq 0$, $\exists\ a \in A$, $(x'_i,x'_{i+1}) \in R_a$, and (4) $\forall\ i \geq 0$, $x'_i \in X_1$. Thus x_1 is a sequential vertex for X_1. Since X_1 is arbitrary, G must be sequential.

Q.E.D.

V. BIBLIOGRAPHY

1. Berge, Claude. The Theory of Graphs and Its Applications. John Wiley, 1962.

2. Birkhoff, G. Lattice Theory. AMS Colloquium Publication, 28, 1948.

3. Cohn, P. M. Universal Algebra. Harper and Row, 1965.

4. Gimpel, J. F. "A Reduction Technique for Prime Implicant Tables." Switching Circuit Theory and Logical Design, pp. 183-191, Oct. 1964.

5. Ginsburg, S. An Introduction to Mathematical Machine Theory. Addison-Wesley, 1962.

6. Hartmanis, J. and Stearns, R. E. "Some Danger in State Reduction of Sequential Machine," Information and Control, 5, pp. 252-260, Sept. 1962.

7. Hartmanis, J. and Stearns, R. E. "Pair Algebra and Its Application to Automata Theory," Information and Control, 7, pp. 485-507, Dec. 1964.

8. Hohn, F. E. and Schissler, L. R. "Boolean Matrices and the Design of Combinational Relay Circuits," Bell System Technical Journal, vol. 34, pp. 177-202, Jan. 1955.

9. McCluskey, E. J. Introduction to the Theory of Switching Circuits. McGraw-Hill, 1965.

10. Muller, D. E. "A Treatment of Sequential Machines," File No. 567, Dept. of Computer Science, University of Illinois

11. Muller, D. E. "Lecture Notes on Automata Theory." University of Illinois, Sept. 1965.

12. Pickett, H. E. "Direct and Subdirect Product: Structure Theorems for Non-Deterministic Automata." Chevon Research Company, 1966.

13. Wensel, G. "Report on the paper by M. Yoeli: Multi-Valued Homomorphism and Subdirect Covers of Partial Algebras."

APPENDIX

The concepts of weak congruence relations, covers with S.P., and homomorphic relations on graphs have been shown to be closely related. The intention here is to state briefly how these concepts arise in state reduction and decomposition of sequential machines.

Definition A. A sequential machine M is a 5-tuple $\langle A,B,Q,f,g\rangle$ such that A, B and Q are finite sets and $f \subseteq (Q \times A) \times Q$ and $g \subseteq (Q \times A) \times B$.

It is well known that every sequential machine $M = \langle A,B,Q,F,g\rangle$ can be represented by a graph $G(M) = \langle Q,A\rangle$ such that $\forall\, a \in A$, $q \in (p,a) \cdot f \rightleftarrows (p,q) \in R_a$. We note that M is complete iff G(M) is productive, and M is deterministic iff $\forall\, a \in A$, R_a is single-valued. When M is deterministic, we will use the convention that $\forall\, a \in A$, $\forall\, q \in Q$, $(q,a) \cdot f = qa$. In the following, only incomplete, deterministic sequential machines will be considered.

Definition B. Let $M = \langle A,B,Q,f,g\rangle$ be a sequential machine and A^* the free semi-group generated by A under concatenation. Then $\forall\, x = a_1 \ldots a_n \in A^*$, $\forall\, q \in Q$, $(q,x) \cdot g$ exists iff each state $q_i a_i$, $1 \leq i \leq n$ and each output $(q_i,a_i) \cdot g$, $1 \leq i \leq n$ exist; and when $(q,x) \cdot g$ exists, it is defined to be the sequence $(q_1,a_1) \cdot g$, ..., $(q_n,a_n) \cdot g$. where $q_1 = q$ and $q_{i+1} = q_i \cdot a_i$ for $i \geq 1$.

Definition C. Let $M = \langle A,B,Q,f,g\rangle$ be a sequential machine. $\forall\, x \in A^*$, $\forall\, q \in Q$, x is applicable to q iff $(q,x) \cdot g$ exists.

Definition D. Two states p,q in $M = \langle A,B,Q,f,g\rangle$ is <u>compatible</u> iff $\forall\, x \in A^*$ applicable to both p and q, $(p,x)\cdot g = (q,x)\cdot g$.

The state reduction problem of a machine $M = \langle A,B,Q,f,g\rangle$ is essentially a problem of finding a cover with S.P. $C = \{X_i\}_{i\in I}$ on Q such that $\forall\, i \in I$, $\forall\, p,q \in X_i$, p and q are compatible. We will show in the following that compatibility on Q is a weak congruence relation of type II on M. Define $\theta_1 \in Q \times Q$ such that $\forall\, p,q \in Q$, $(p,q) \in \theta_1$ iff $\forall\, a \in A$ applicable to both p and q, $(p,a) \cdot g = (q,a) \cdot g$. Assume θ_k has been defined $\forall\, k \geq 1$. Define θ_{k+1} on Q by the rule:

$$\forall\, p,q \in Q,\ (p,q) \in \theta_{k+1} \rightleftarrows (p,q) \in \theta_k \wedge \forall\, a \in A,\ (p\cdot a, q\cdot a) \in \theta_k.$$

If $|Q| = n$, then define θ such that

$$\theta = \theta_{\frac{n(n-1)}{2}}$$

It can be shown readily that $\theta = \delta(\theta_1)$ and that $\forall\, p,q \in Q$, $(p,q) \in \theta \rightleftarrows p$ and q are compatible. Now, if C is any irredundant cover with S.P. on Q such that $\theta_C = \theta$. Then M/C is a reduced machine of M.

Definition E. Let $M = \langle A,B,Q,f,g\rangle$ be a machine and $C = \{X_i\}_{i\in I}$ a cover with S.P. on Q. Then $\forall\, q \in Q$, the <u>degree of q with respect to C</u>, $d(q)$, is defined by

$$d(q) = \left|\ \{X_i \mid X_i \in C \wedge q \in X_i\}\ \right| .$$

In studying the realizability of a given machine by smaller machines, the concept of "state splitting" arises when a given machine is indecomposable or does not have economical realizations. It was found that enlarging the given machine by adding redundant states (state splitting) may lead to desirable realization. More specifically, the state splitting operation on a machine $M = \langle A,B,Q,f,g\rangle$ is a multi-valued mapping $H: Q \to Q'$ such that $\forall\, q \in Q$, $q \cdot H = \{q,q_1,\ldots,q_{d(q)-1}\}$. (I.e., embedding G(M) into a larger graph G(M')). Then a cover with S.P. C on M induces a partition C' on $M' = \langle A,B,Q',f',g'\rangle$ such that elements of C and C' correspond one to one. It can be shown that with suitable definition of f', H is indeed a (strong) homomorphic relation. We note that, however, f' is not unique.

VITA

The author, Raymond Tzuu-yau Yeh, was born in Hunan, China on November 5, 1937. He received his B.S. degree in Electrical Engineering in June, 1961, and his M.A. degree in Mathematics in February, 1963 from the University of Illinois. From September 1961 to August 1963, he was a research assistant in Biological Computer group of the Electrical Engineering Research Laboratory of the University of Illinois. Since September, 1963, he has been a research assistant in the Department of Computer Science.

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